Haar wavelet method for solving some nonlinear Parabolic equations

G. Hariharan · K. Kannan

Abstract   Wavelet transform or wavelet analysis is a recently developed mathematical tool in applied mathematics. In this paper, we develop an accurate and efficient Haar transform or Haar wavelet method for some of the well-known nonlinear parabolic partial differential equations. The equations include the Nowell-whitehead equation, Cahn-Allen equation, FitzHugh-Nagumo equation, Fisher's equation, Burger's equation and the Burgers-Fisher equation. The proposed scheme can be used to a wide class of nonlinear equations. The power of this manageable method is confirmed. Moreover the use of Haar wavelets is found to be accurate, simple, fast, flexible, convenient, small computation costs and computationally attractive.

Keywords   Nonlinear parabolic equations · Haar wavelet method

1 Introduction

Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics, optical fibers, biology, fluid dynamics and chemical kinetics. The concepts like solitons, peakons, kinks, breathers, cusps and compactons are now thoroughly investigated in the scientific literature [1,2]. A variety of powerful methods, such as inverse scattering method [3], bilinear transformation [4], Backland transformation, a bilinear form [4], the tanh-sech method [5–7], extended tanh method [6], sine-cosine method [8], homogeneous balance method [9], Exp-function method [10], the tanh method [11], Adomian decomposition method [12], the tanh-coth method [13], Jacobi elliptic functions, and a Lax pair [14] have been used independently by which soliton and multi-soliton solutions are obtained. Recently, Haar wavelets have been applied extensively for signal processing

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in communications and research area in Physics, and have proved to be a wonderful mathematical tool. Haar wavelet method is used to solve some of the nonlinear PDEs given by

$$u_t = u_{xx} + \alpha u + \beta u^n, \tag{1}$$

$$u_t = u_{xx} + \alpha + \beta e^{n\lambda u}, \tag{2}$$

$$u_t = u_{xx} - u(1-u)(\alpha - u), \tag{3}$$

$$u_t = u_{xx} + \alpha u u_x, \tag{4}$$

and

$$u_t = u_{xx} + \alpha u u_x + k u(1-u) \tag{5}$$

where α, β, k and λ are constants.

In solving ordinary differential equations by using Haar wavelet related method, Chen and Hsiao [15] had derived an operational matrix of integration based on Haar wavelet. Lepik [16–18] had solved higher order as well as nonlinear ODEs and some nonlinear evolution equations by Haar wavelet method. Hariharan et al. [19] have introduced the solution of Fisher's equation by Haar wavelet method.

Equation (1) gives rise to three known models. For $\alpha = 1$, $\beta = -1$ and $n = 3$, Eq. (1) becomes the Cahn-Allen equation. It arises in many scientific applications such as mathematical biology, quantum mechanics and plasma physics. It is well known that wave phenomena of plasma media and fluid dynamics are modeled by kink shaped and tanh solution or bell shaped sech solutions. The Cahn-Allen equation serves as a model for the study of phase separation in isothermal, isotropic, binary mixtures such as molten alloys. If $n = 3$ and the coefficient β is replaced by $-\beta$, then Eq. (1) becomes the Newell-Whitehead equation. The Newell-Whitehead equation describes the dynamical behavior near the bifurcation point for the Rayleigh-Benard convection of binary fluid mixtures [20].

For $n = 2$ and $\beta = -\alpha$, Eq. (1) reduces to the well-known Fisher's equation. The Fisher's equation combines diffusion with logistic nonlinearity. This equation is encountered in chemical kinetics and population dynamics, which includes problems such as nonlinear evolution of a population in a one-dimensional habitat, neutron population in a nuclear reaction. Moreover, the same equation occurs in logistic population growth models [21], flame propagation, neurophysiology, autocatalytic chemical reactions, and branching Brownian motion processes. The mathematical properties of Fisher's Equation (FE) have been studied extensively and there have been numerous discussions in the literature.

Equation (2) is a parabolic equation with exponential nonlinearity. Moreover, Eqs. (3–5) represent the FitzHugh- Nagumo equation, the Burger's equation and the Burgers-Fisher equation respectively. The FitzHugh- Nagumo equation describes the dynamical behavior near the bifurcation point for the Rayleigh-Benard convection of binary fluid mixtures [20]. It is an important nonlinear reaction–diffusion equation and applied to model the transmission of nerve impulses, also used in biology and the area of population genetics, in circuit theory. The Burgers' equation, which is a nonlinear

partial differential equation of second order, is used in disciplines as a simplified model for turbulence, boundary layer behavior, shock wave formation and mass transport. The equation serves as a nonlinear analog of the fluid mechanics equations because it has terms, which closely duplicate the physical properties, i.e., a convective term, a diffusive term, and a time dependent term. Nonlinear Partial differential equations (NLPDEs) arise in many fields of science, particularly in physics, engineering, chemistry and finance, and are fundamental for the mathematical formulation of continuum models. The distinctive feature of the Burgers equation is that it is the simplest mathematical formulation of the competition between convection and diffusion. Another feature of the Burgers equation is that although it does not have a pressure gradient term it still is a good approximation of the propagation of one-dimensional disturbances. Performance of a numerical method can be judged from its ability to resolve the large gradient region that develops in the solution. Many problems can be modeled by the Burger's equation. For example, the Burger's equation can be considered as an approach to the Navier-Stokes equation since both contains nonlinear term of the type: unknown functions multiplied by a first derivative and they contain higher-order terms multiplied by a small parameter.

Equations (1–5) arise in many scientific applications such as mathematical biology, quantum mechanics and plasma physics. Ablowitz and Segur [3] have implemented the inverse scattering transform method to handle the nonlinear equations of physical significance. Mansour [22] showed that traveling wave solutions of a nonlinear reaction-diffusion-chemotaxis model for bacterial pattern formation. Olmos and Shizgal [23] have showed a pseudo-spectral method for solving Fisher's equation. Wazwaz [24] obtained the analytical solutions of the Burgers, Fisher, and Huxley equations and combined forms of these equations. Rajendran and Senthamarai [25] have established traveling wave solution of nonlinear coupled reaction-diffusion equation arising in mathematical chemistry. Kolmogorov et al. [26] showed that any initial concentration which is one for large negative spatial variable x and vanishes for large x, evolves to a traveling wave front with minimal velocity $v = 2\sqrt{k_0}$. Different initial values propagate with different traveling waves, depending on the behavior at $x \to \pm\infty$.

We introduce a Haar wavelet method for solving the Equations (1–5) with the initial and boundary conditions, which will exhibit several advantageous features:

(1) Very high accuracy fast transformation and possibility of implementation of fast algorithms compared with other known methods.
(2) The simplicity and small computation costs, resulting from the sparsity of the transform matrices and the small number of significant wavelet coefficients.
(3) The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically.

Beginning from 1980's, wavelets have been used for solution of partial differential equations (PDE). The good features of this approach are possibility to detect singularities, irregular structure and transient phenomena exhibited by the analyzed equations. Most of the wavelet algorithms can handle exactly periodic boundary conditions. The wavelet algorithms for solving PDE are based on the Galerkin techniques or on the collocation method.

Evidently all attempts to simplify the wavelet solutions for PDE are welcome. One possibility for this is to make use of the Haar wavelet family. Haar wavelets (which are Daubechies of order 1) consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. A drawback of the Haar wavelets is their discontinuity. Since the derivatives do not exist in the breaking points it is not possible to apply the Haar wavelets for solving PDE directly. There are two possibilities for getting out of this situation. One way is to regularize the Haar wavelets with interpolating splines (e.g. B-splines or Deslaurier-Dabuc interpolating wavelets). This approach has been applied by Cattani [27]. The other way is to make use of the integral method, which was proposed by Chen and Hsiao [15]. There are some useful discussions by other researchers [28,29].

The advantages of computational and memory requirements of the Haar transform make it use of a considerable interest to VLSI designers as well. The use of this set of CAD tools allowed the derivation of strategies for testing MOS circuits when memory states were encountered as a consequence of some type of faults. The advantage of using Haar functions instead of Walsh functions in CAD system based on spectra methods for some classes of Boolean functions was shown in [30,31]. For example, the analysis in [30] shows that the spectral complexity of conjunction and disjunction increases with the number of variables, exponentially for the Walsh functions and only linearly for the Haar functions. Karpovsky [30] noticed that the size of the memory block can be optimized only when the Haar series is used.

The paper is organized the following way. For completeness sake the Haar wavelet method is presented in Sect. 2. Function approximation is presented in Sect. 3. The method of solution of the general nonlinear parabolic PDE, the parabolic equation with exponential nonlinearity, the Fitzhugh-Nagumo equation, the Burgers' equation and the Burgers-Fisher equation are proposed in Sect. 4. Concluding remarks are given in Sect. 5.

2 Haar wavelets

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar [32]. The Haar transform is one of the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the mean time, several definitions of the Haar functions and various generalizations have been published and used. They were intended to adopt this concept to some practical applications as well as to extend its in applications to different classes of signals. Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design.

After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

The pioneer work in system analysis via Haar wavelets was done by Chen and Hsiao [15], who first derived a Haar operational matrix for the integrals of the Haar
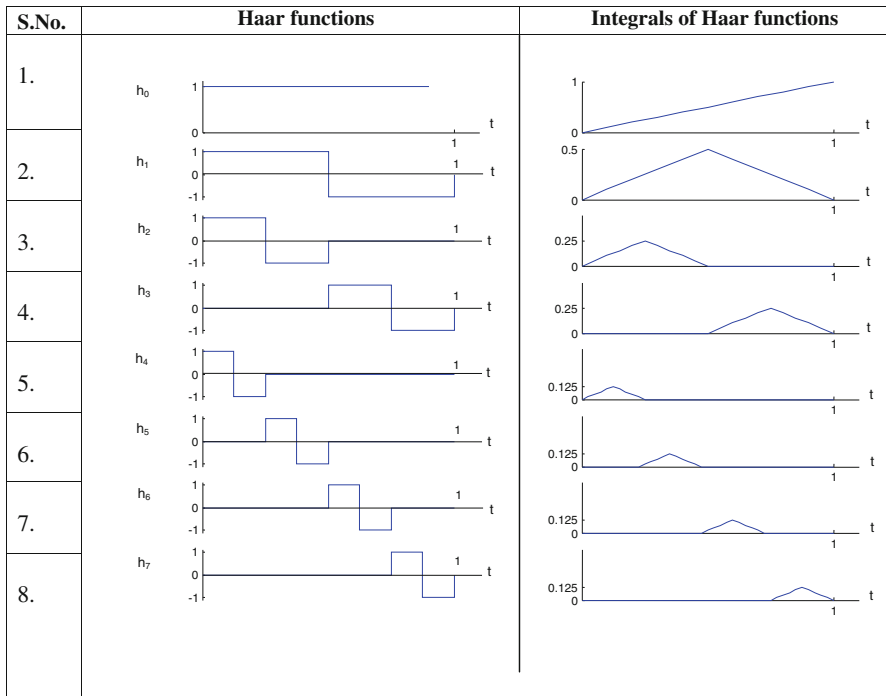
S.No.	Haar functions	Integrals of Haar functions

Fig. 1 First eight Haar functions and their integrals

function vector and paved the way for the Haar analysis of dynamic systems. The fundamental work in states analysis and parameters estimation of bilinear systems via Haar wavelets was done by of Hsiao, who first derived a Haar product matrix and a coefficient matrix. Hsiao and Wang [29] proposed a key idea to transform the time-varying function and its product with the states into a Haar product matrix.

The orthogonal set of Haar function $h_i(x)$ is shown in Fig. 1. This is a group of square waves with magnitudes of ± 1 in certain intervals and zeros elsewhere.

For applications of the Haar transform in logic design, efficient ways of calculating the Haar spectrum from reduced forms of Boolean functions are needed.

The Haar wavelet family for $x \in [0, 1]$ is defined as follows.

$$h_i(x) = \begin{cases} 1, & \text{for } x \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1, & \text{for } x \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0, & \text{elsewhere} \end{cases} \qquad (6)$$

Integer $m = 2^j (j = 0, 1, 2, \ldots, J)$ indicates the level of the wavelet; $k = 0, 1, 2, \ldots, m - 1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m + k + 1$; in case of minimal values. $m = 1, k = 0$ we have $i = 2$, the maximal value of i is $2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 \equiv 1\ in$ [0, 1]. Let us

define the collocation points $x_l = (l - 0.5)/2M, (l = 1, 2, \ldots, 2M)$ and discretise the Haar function $h_i(x)$; in this way we get the coefficient matrix $H(i, l) = (h_i(x_l))$, which has the dimension $2M \times 2M$.

In the wavelet analysis for a dynamic system, all functions need to be transformed into Haar series. Since the differentiation of Haar wavelets always results in impulse functions, which should be avoided, the integration of Haar wavelets is preferred, which should be expandable into Haar series with Haar coefficient matrix P [15].

It is assumed that the value $i = 1$ corresponds to the scaling function for which

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [-1, 1), \\ 0 & \text{elsewhere} \end{cases}$$

The operational matrix of integration P, which is a $2M$ square matrix, is defined by the equation

$$P_i(x) = \int_0^x h_i(x)dx, \quad x \in [0, 1), \tag{7}$$

In Sect. 4 the second order differential equation was replaced with of system of first order equations. Here another variant of solution is demonstrated, by which the second order equation is directly solved. For this purpose the integral operator Q is introduced.

$$Q_i(x) = \int_0^x P_i(x)\,dx \tag{8}$$

This integral can be evaluated according to (6). For instance for $2M = 4$ we have

$$QH = \frac{1}{128} \begin{bmatrix} 1 & 9 & 25 & 49 \\ 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

The elements of the matrices H, P and Q can be evaluated according to (6), (7) and (8).

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, P_2 = \frac{1}{4} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad P_4 = \frac{1}{16} \begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$P_8 = \frac{1}{64} \begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chen and Hsiao [15] derived that the following matrix equation for calculating the matrix P of order m holds

$$P_{(m)} = \frac{1}{2m} \begin{pmatrix} 2m\, P_{(m/2)} & -H_{(m/2)} \\ H_{(m/2)}^{-1} & O \end{pmatrix}$$

where O is a null matrix of order $\frac{m}{2} \times \frac{m}{2}$,

$$H_{(m \times m)} \triangleq \big[h_m(x_0), h_m(x_1), \ldots, h_m(x_{m-1}) \big] \tag{9}$$

and $\frac{i}{m} \leq t < \frac{(i+1)}{m}$ and $H_{(m \times m)}^{-1} = \frac{1}{m} H_{(m \times m)}^T diag(r)$

Here a diagonal matrix $diag\,(r)$ is defined as $diag\,(r) \triangleq \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}$, r is the

row vector (or column vector) with elements $r_1, r_2, \ldots, r_n$.

It should be noted that calculations for $P_{(m)}$ and $H_{(m)}$ must be carried out only once; after that they will be applicable for solving whatever differential equations.

The fast capability of HT should be impressive. Since H and H^{-1} contain many zeros, this phenomenon makes the Haar transform faster than the Fourier transform, and it is even faster than the Walsh transform. This is one of the reasons for rapid convergence of the Haar wavelet series. The numbers of additions and multiplications for these three transforms are shown in Table 1.

In practical applications, a small number of terms increases the calculation speed and saves memory storage; a large number of terms improve resolution accuracy. Therefore, a trade-off between calculation speed, memory saving, and the resolution accuracy has been considered in the analysis.

Table 1 Comparison of algorithmic complexity of the of the proposed method with FFT and WT

Series	Numbers of additions	Numbers of multiplications
Haar transform (HT)	$2m - 2$	m
Walsh transform (WT)	$m \log_2 m$	m
Fast fourier transform (FFT)	$m \log_2 m$	$m\,(\log_2 m + 1)$

3 Function approximation

Any square integrable function $y(x)$ in the interval $[0, 1)$ can be expanded by a Haar series of infinite terms

$$y(x) = \sum_{i=0}^{\infty} c_i h_i(x), \quad i \in \{0\} \cup N, \tag{10}$$

where the Haar coefficients c_i are determined as,

$$c_0 = \int_0^1 y(x) h_0(x)\, dx, \quad c_n = 2^j \int_0^1 y(x) h_i(x) dx,$$

$$i = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j, \quad x \in [0, 1) \tag{11}$$

such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 \left[y(x) - \sum_{i=0}^{m-1} c_i h_i(x) \right]^2 dx, \quad m = 2^j, \quad j \in \{0\} \cup N.$$

Usually, the series expansion of (11) contains infinite terms for smooth $y(x)$. If $y(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite m terms, that is

$$y(x) = \sum_{i=0}^{m-1} c_i h_i(x) = c_{(m)}^T h_{(m)}(x) \tag{12}$$

where the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as

$$c_{(m)}^T = [c_0, c_1, \ldots, c_{m-1}]$$

and $h_{(m)}(x) = [h_0(x), h_1(x), \ldots, h_{m-1}(x)]^T$ where 'T' means transpose and $m = 2^j$.

4 The general nonlinear parabolic PDEs

The general nonlinear parabolic equation is of the form

$$u_t = u_{xx} + \alpha u + \beta u^n \tag{13}$$

with the initial condition $u(x, 0) = f(x), 0 \leq x \leq 1$ and the boundary conditions $u(0, t) = g_0(t), u(1, t) = g_1(t), 0 < t \leq T$

Let us divide the interval (0,1] into N equal parts of length $\Delta t = (0, 1]/N$ and denote $t_s = (s - 1)\Delta t, s = 1, 2, \ldots, N$. We assume that $\dot{u}''(x, t)$ can be expanded interms of Haar wavelets as formula

$$\dot{u}''(x, t) = \sum_{i=0}^{m-1} c_s(i) h_i(x) = c_{(m)}^T h_{(m)}(x) \tag{14}$$

where $\cdot$ and $'$ means differentiation with respect to t and x respectively, the row vector $c_{(m)}^T$ is constant in the subinterval $t \in (t_s, t_{s+1}]$

Integrating formula (14) with respect to t from t_s to t and twice with respect to x from 0 to x, we obtain

$$u''(x, t) = (t - t_s)c_{(m)}^T h_{(m)}(x) + u''(x, t_s) \tag{15}$$

$$u(x, t) = (t - t_s)c_{(m)}^T Q_{(m)}h_{(m)}(x)$$
$$+u(x, t_s) - u(0, t_s) + x[u'(0, t) - u'(0, t_s)] + u(0, t) \tag{16}$$

$$\dot{u}(x, t) = c_{(m)}^T Q_{(m)}h_{(m)}(x) + x\dot{u}'(0, t) + \dot{u}(0, t) \tag{17}$$

With the boundary conditions, we obtain

$$u(0, t_s) = g_0(t_s), \quad u(1, t_s) = g_1(t_s)$$
$$\dot{u}(0, t) = g_0'(t), \quad \dot{u}(1, t) = g_1'(t)$$

Putting $x = 1$ in formulae (16) and (17), we have

$$u'(0, t) - u'(0, t_s) = -(t - t_s)c_{(m)}^T P_{(m)}h_{(m)}(x) + g_1(t) - g_0(t)$$
$$-g_1(t_s) + g_0(t_s) \tag{18}$$

$$\dot{u}'(0, t) = g_1'(t) - c_{(m)}^T Q_{(m)}h_{(m)}(x)f - g_0'(t) \tag{19}$$

where the vector f is defined as

$$f = [1, \underbrace{0, \ldots, 0}_{(m-1)elements}]^T$$

Substituting formulae (18) and (19) into formulae (15–17), and discretizising the results by assuming $x \to x_l, t \to t_{s+1}$ we obtain

$$u''(x_l, t_{s+1}) = (t_{s+1} - t_s)c_{(m)}^T h_{(m)}(x_l) + u''(x_l, t_s) \tag{20}$$

$$u(x_l, t_{s+1}) = (t_{s+1} - t_s)c_{(m)}^T Q_{(m)}h_{(m)}(x_l) + u(x_l, t_s) - g_0(t_s) + g_0(t_{s+1})$$
$$+x_l[-(t_{s+1} - t_s)c_{(m)}^T P_{(m)}f + g_l(t_{s+1})$$
$$-g_0(t_{s+1}) - g_1(t_s) + g_0(t_s)] \tag{21}$$

$$\dot{u}(x_l, t_{s+1}) = c_{(m)}^T Q_{(m)} h_{(m)}(x) + g_0'(t_{s+1})$$
$$+ x_l [-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] \tag{22}$$

There are several possibilities for treating the nonlinearity in Eq. (13). In the following the scheme

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) + \alpha u(x_l, t_{s+1}) + \beta u^n(x_l, t_{s+1}) \tag{23}$$

which leads us from the time layer t_s to t_{s+1} is used.

Substituting formulae (20–22) into the formula (23), we gain

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l [-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1})$$
$$= u''(x_l, t_{s+1}) + \alpha u(x_l, t_{s+1}) + \beta u^n(x_l, t_{s+1}) \tag{24}$$

From formula (24) the wavelet coefficients $c_{(m)}^T$ can be successively calculated.

For clarity in the illustrative example, we assume that $n = 2$ and $\beta = -\alpha$.

Equation (1) reduces to the well-known Fisher's equation. The Fisher's equation combines diffusion with logistic nonlinearity.

$$u_t = u_{xx} + \alpha u (1 - u), \quad 0 < x < 1$$

subject to the initial condition

$$u(x, 0) = \frac{1}{(1 + e^x)^2}.$$

The exact solution in a closed form is given by

$$u(x, t) = \frac{1}{\left(1 + e^{\left(\sqrt{\frac{\alpha}{6}}\right)x - \frac{5}{6}\alpha t}\right)^2}$$

In the following the scheme

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) + \alpha u(x_l, t_{s+1}) \left[1 - u(x_l, t_{s+1})\right]$$

which leads us from the time layer t_s to t_{s+1} is used.

Substituting formulae (20–22) into the above formula, we gain

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l [-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1})$$
$$= u''(x_l, t_{s+1}) + \alpha u(x_l, t_{s+1}) \left[1 - u^n(x_l, t_{s+1})\right]$$

From the above formula the wavelet coefficients $c_{(m)}^T$ can be successively calculated.
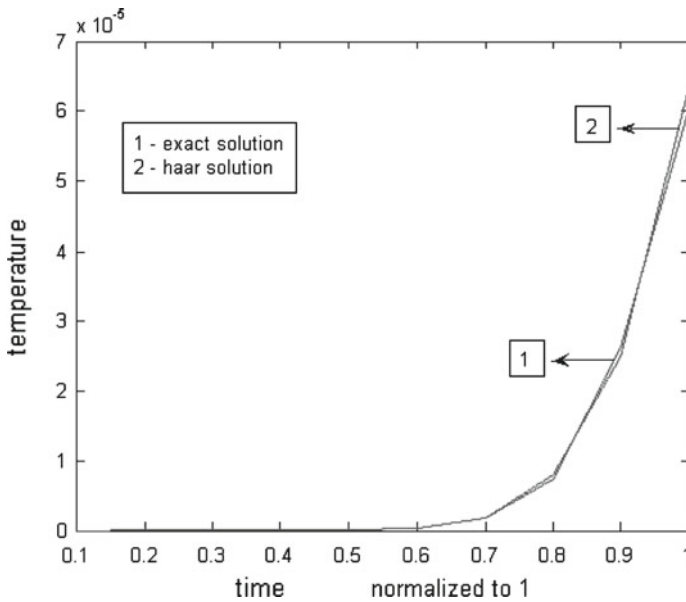
Fig. 2 Comparison between exact and Haar solution of Fisher's equation for $x = 5$ and $m = 16$

This process is started with

$$u(x_l, t_s) = \frac{1}{(1 + e^x)^2},$$

$$u'(x_l, t_s) = \frac{-2e^x}{(1 + e^x)^3},$$

$$u''(x_l, t_s) = -2\left[\frac{e^x - 2e^{2x}}{(1 + e^x)^4}\right].$$

It is worth noting that applying the scheme proposed above for the Fisher's equation $u_t = u_{xx} + \alpha u(1 - u)$, the solution $u(x, t) = \frac{1}{\left(1 + e^{\left(\sqrt{\frac{\alpha}{6}}\right)x - \frac{5}{6}\alpha t}\right)^2}$ can be compared with the Haar solution.

The Haar solution and the analytical solution have been drawn together in the Figs. 2 and 3 for space variable $x = 5, 20$ and for $t \in [0, 1)$. Computer simulation was carried out in the case of $m = 16$. The computed results were compared with the exact solution. More accurate results can be obtained by using larger m. Comparison with these algorithm shows that the Haar wavelet method is competitive and efficient. The advantages of our method are its simplicity and speed of convergence, which is caused from the sparseness of the transform matrices H, P, Q.

In Fig. 4a, the good agreement between the numerical results and the analytic solution with $\alpha = 2000$ and 40 collocation points is shown. In order to get a stable and
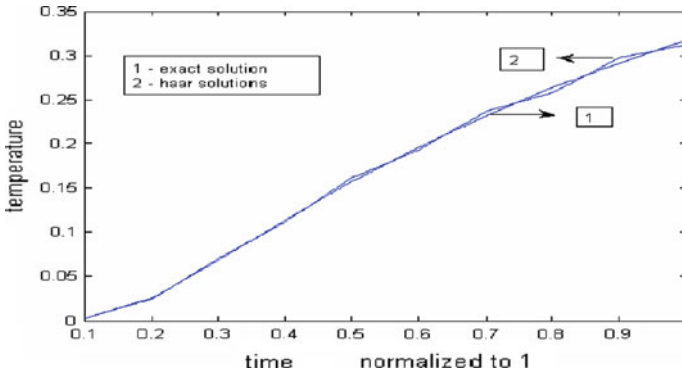
Fig. 3 Comparison between exact and Haar solution of Fisher's equation for $x = 20$ and $m = 16$
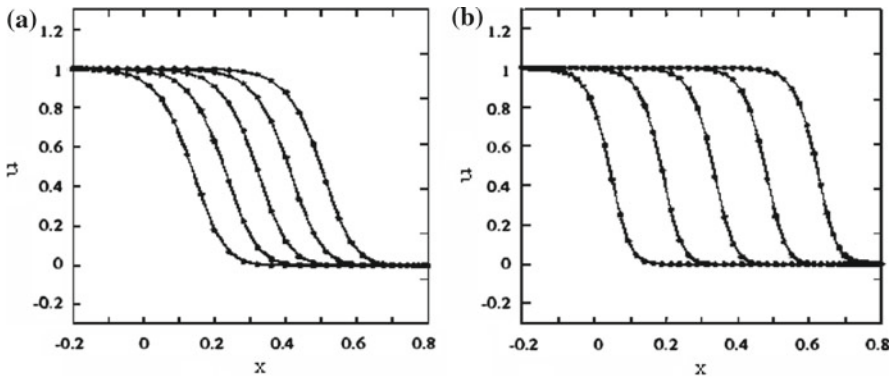

Fig. 4 Time dependent profiles versus x for α and for times **(a)** $t = 0.002, 0.003, 0.004, 0.005, 0.006$; **(b)** $t = 0.0005, 0.0015, 0.0025, 0.0035, 0.0045$

accurate solution for $\alpha = 5000$ shown in Fig. 4b, the number of points had to be increased from 40 to 64 collocation points.

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System with Processor Intel(R) Core$^{(TM)}$ 2 Duo CPU T5470 @ 1.60 GHz(2CPUs) and 1 GB RAM.

4.1 Parabolic equation with exponential nonlinearity

Consider the following parabolic equation with exponential nonlinearity:

$$u_t = u_{xx} + \alpha + \beta e^{n\lambda u} \tag{25}$$

with the initial condition $u(x, 0) = f(x), 0 \leq x \leq 1$ and the boundary conditions $u(0, t) = g_0(t), u(1, t) = g_1(t), 0 < t \leq T$

Let us divide the interval (0,1] into N equal parts of length $\Delta t = (0, 1]/N$ and denote $t_s = (s - 1)\Delta t, s = 1, 2, \ldots, N$. We assume that $\dot{u}''(x, t)$ can be expanded interms of Haar wavelets as formula

$$\dot{u}''(x, t) = \sum_{i=0}^{m-1} c_s(i) h_i(x) = c_{(m)}^T h_{(m)}(x) \tag{26}$$

where. and $\prime$ means differentiation with respect to t and x respectively, the row vector $c_{(m)}^T$ is constant in the subinterval $t \in (t_s, t_{s+1}]$

Integrating formula (26) with respect to t from t_s to t and twice with respect to x from 0 to x, we obtain

$$u''(x, t) = (t - t_s) c_{(m)}^T h_{(m)}(x) + u''(x, t_s) \tag{27}$$

$$u(x, t) = (t - t_s) c_{(m)}^T Q_{(m)} h_{(m)}(x) + u(x, t_s) - u(0, t_s)$$
$$+ x[u'(0, t) - u'(0, t_s)] + u(0, t) \tag{28}$$

$$\dot{u}(x, t) = c_{(m)}^T Q_{(m)} h_{(m)}(x) + x\dot{u}'(0, t) + \dot{u}(0, t) \tag{29}$$

Using the boundary conditions, we obtain

$$u(0, t_s) = g_0(t_s), \quad u(1, t_s) = g_1(t_s)$$
$$\dot{u}(0, t) = g_0'(t), \quad \dot{u}(1, t) = g_1'(t)$$

Putting $x = 1$ in formulae (28) and (29), we have

$$u'(0, t) - u'(0, t_s) = -(t - t_s) c_{(m)}^T P_{(m)} h_{(m)}(x)$$
$$+ g_1(t) - g_0(t) - g_1(t_s) + g_0(t_s) \tag{30}$$

$$\dot{u}'(0, t) = g_1'(t) - c_{(m)}^T Q_{(m)} h_{(m)}(x) f - g_0'(t) \tag{31}$$

where the vector f is defined as

$$f = [1, \underbrace{0, \ldots, 0}_{(m-1) elements}]^T$$

Substituting formulae (30) and (31) into formulae (27–29), and discretizising the results by assuming $x \to x_l, t \to t_{s+1}$ we obtain

$$u''(x_l, t_{s+1}) = (t_{s+1} - t_s) c_{(m)}^T h_{(m)}(x_l) + u''(x_l, t_s) \tag{32}$$

$$u(x_l, t_{s+1}) = (t_{s+1} - t_s) c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + u(x_l, t_s) - g_0(t_s) + g_0(t_{s+1})$$
$$+ x_l[-(t_{s+1} - t_s) c_{(m)}^T P_{(m)} f + g_1(t_{s+1})$$
$$- g_0(t_{s+1}) - g_1(t_s) + g_0(t_s)] \tag{33}$$

$$\dot{u}(x_l, t_{s+1}) = c_{(m)}^T Q_{(m)} h_{(m)}(x) + g_0'(t_{s+1}) + x_l[-c_{(m)}^T P_{(m)} f$$
$$+ g_1'(t_{s+1}) - g_0'(t_{s+1})] \tag{34}$$

There are several possibilities for treating the nonlinearity in Eq. (25). In the following the scheme

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) + \alpha + \beta e^{n\lambda u(x_l, t_{s+1})} \tag{35}$$

which leads us from the time layer t_s to t_{s+1} is used.

Substituting formulae (32–34) into the formula (35), we gain

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l[-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1})$$
$$= u''(x_l, t_{s+1}) + \alpha + \beta e^{n\lambda u(x_l, t_{s+1})} \tag{36}$$

From formula (36) the wavelet coefficients $c_{(m)}^T$ can be successively calculated.

4.2 The FitzHugh-Nagumo equation

We consider the FitzHugh-Nagumo equation

$$u_t = u_{xx} - u(1 - u)(\alpha - u) \tag{37}$$

with the initial condition $u(x, 0) = f(x), 0 \leq x \leq 1$ and the boundary conditions $u(0, t) = g_0(t), u(1, t) = g_1(t), 0 < t \leq T$

A similar procedure applied in Sect. 4.1 leads to the following scheme

$$u''(x_l, t_{s+1}) = (t_{s+1} - t_s) c_{(m)}^T h_{(m)}(x_l) + u''(x_l, t_s) \tag{38}$$

$$u(x_l, t_{s+1}) = (t_{s+1} - t_s) c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + u(x_l, t_s) - g_0(t_s) + g_0(t_{s+1})$$
$$+ x_l[-(t_{s+1} - t_s) c_{(m)}^T P_{(m)} f + g_l(t_{s+1})$$
$$- g_0(t_{s+1}) - g_1(t_s) + g_0(t_s)] \tag{39}$$

$$\dot{u}(x_l, t_{s+1}) = c_{(m)}^T Q_{(m)} h_{(m)}(x) + g_0'(t_{s+1}) + x_l[-c_{(m)}^T P_{(m)} f$$
$$+ g_1'(t_{s+1}) - g_0'(t_{s+1})] \tag{40}$$

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) - u(x_l, t_{s+1})[1 - u(x_l, t_{s+1})][\alpha - u(x_l, t_{s+1})] \tag{41}$$

Substituting formulae (38–40) into the formula (41), we obtain

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l[-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1})$$
$$= u''(x_l, t_{s+1}) - u(x_l, t_{s+1})[1 - u(x_l, t_{s+1})][\alpha - u(x_l, t_{s+1})] \tag{42}$$

From formula (42) the wavelet coefficients $c_{(m)}^T$ can be successively calculated.

The results in the present paper for Fisher's equation, a prototypical reaction–diffusion equation, will play an important role in the application of Haar wavelet methods to more complex physical systems such as the Fitzhugh–Nagumo equation [33].

4.3 The Burgers' equation

We consider the Burgers' equation

$$u_t = u_{xx} + \alpha u u_x \tag{43}$$

with the initial condition $u(x, 0) = f(x), 0 \le x \le 1$ and the boundary conditions $u(0, t) = g_0(t), u(1, t) = g_1(t), 0 < t \le T$

A similar procedure applied in Sect. 4.1 leads to the following scheme

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) + \alpha u(x_l, t_{s+1}) u'(x_l, t_{s+1}) \tag{44}$$

Then

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l[-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1})$$
$$= u''(x_l, t_{s+1}) + \alpha u(x_l, t_{s+1}) u'(x_l, t_{s+1}) \tag{45}$$

From formula (45) the wavelet coefficients $c_{(m)}^T$ can be successively calculated.

For clarity in the illustrative example, we consider the Burgers' equation

$$u_t + u u_x = v u_{xx}, \quad x \in [0, 1], \quad t > 0,$$

subject to $u(x, 0) = \sin 2\pi x, u(0, t) = u(1, t) = 0.$

The exact solution in a closed form is given by

$$u(x, t) = \frac{2\pi v \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 vt} n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 vt} \cos(n\pi x)}$$

In the following the scheme

$$\dot{u}(x_l, t_{s+1}) + u(x_l, t_{s+1}) u'(x_l, t_{s+1}) = v u''(x_l, t_{s+1})$$

which leads us from the time layer t_s to t_{s+1} is used.

The Haar wavelet scheme is given by

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l[-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1})$$
$$= v u''(x_l, t_{s+1}) - u(x_l, t_{s+1}) u'(x_l, t_{s+1})$$

From the above formula the wavelet coefficients $c_{(m)}^T$ can be successively calculated.

This process is started with

$$u(x_l, t_s) = \sin[2\pi x(l)],$$
$$u'(x_l, t_s) = 2\pi \cos[2\pi x(l)],$$
$$u''(x_l, t_s) = -4\pi^2 \sin[2\pi x(l)].$$

It is worth noting that applying the scheme proposed above for the Burgers' equation $u_t + uu_x = vu_{xx}$, the exact solution $u(x, t) = \frac{2\pi v \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 vt} n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 vt} \cos(n\pi x)}$ can be compared with the Haar solution.

More accurate results can be obtained by using larger m. Comparison with these algorithms shows that the Haar wavelet method is competitive and efficient.

4.4 The Burgers-Fisher equation

We consider the Burgers-Fisher equation

$$u_t = u_{xx} + \alpha uu_x + ku(1 - u) \tag{46}$$

with the initial condition $u(x, 0) = f(x), 0 \leq x \leq 1$ and the boundary conditions $u(0, t) = g_0(t), u(1, t) = g_1(t), 0 < t \leq T$

A similar procedure applied in Sect. 4.1 leads to the following scheme

$$u''(x_l, t_{s+1}) = (t_{s+1} - t_s)c_{(m)}^T h_{(m)}(x_l) + u''(x_l, t_s) \tag{47}$$

$$u'(x_l, t_{s+1}) = (t_{s+1} - t_s)c_{(m)}^T P_{(m)} h_{(m)}(x_l) + u'(x_l, t_s) - (t_{s+1} - t_s)c_{(m)}^T P_{(m)} f \tag{48}$$

$$u(x_l, t_{s+1}) = (t_{s+1} - t_s)c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + u(x_l, t_s) - g_0(t_s) + g_0(t_{s+1})$$
$$+ x_l[-(t_{s+1} - t_s)c_{(m)}^T P_{(m)} f + g_l(t_{s+1}) - g_0(t_{s+1})$$
$$- g_1(t_s) + g_0(t_s)] \tag{49}$$

$$\dot{u}(x_l, t_{s+1}) = c_{(m)}^T Q_{(m)} h_{(m)}(x) + g_0'(t_{s+1}) + x_l[-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1})$$
$$- g_0'(t_{s+1})] \tag{50}$$

In the following the scheme

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) + \alpha u(x_l, t_{s+1})u'(x_l, t_{s+1})$$
$$+ ku(x_l, t_{s+1}) \left[1 - u(x_l, t_{s+1})\right] \tag{51}$$

which leads us from the time layer t_s to t_{s+1} is used.

Substituting Eqs. (47–50) into the equation (51), we obtain

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l[-c_{(m)}^T P_{(m)} f + g_1'(t_{s+1}) - g_0'(t_{s+1})] + g_0'(t_{s+1})$$
$$= u''(x_l, t_{s+1}) + \alpha u(x_l, t_{s+1})u'(x_l, t_{s+1}) + ku(x_l, t_{s+1}) \left[1 - u(x_l, t_{s+1})\right] \tag{52}$$

From formula (52) the wavelet coefficients $c_{(m)}^T$ can be successively calculated.

More accurate results can be obtained by using larger m. Comparison with these algorithms shows that the Haar wavelet method is competitive and efficient.

5 Conclusion

The theoretical elegance of the Haar wavelet approach can be appreciated from the simple mathematical relations and their compact derivations and proofs. It has been well demonstrated that in applying the nice properties of Haar wavelets, the differential equations can be solved conveniently and accurately by using Haar wavelet method systematically. According to this method the spatial operators are approximated by the Haar wavelet method and the time derivation operators by the finite difference method The main advantages of this method is its simplicity and small computation costs: it is due to the sparcity of the transform matrices and to the small number of significant wavelet coefficients. In comparison with existing numerical schemes used to solve the nonlinear parabolic equations, the scheme in this paper is an improvement over other methods in terms of accuracy. It is worth mentioning that Haar solution provides excellent results even for small values of m ($m = 16$). For larger values of m (*ie.*, $m = 32, m = 64, m = 128, m = 256$), we can obtain the results closer to the real values. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones.

The main goal of this work is to apply the Haar wavelet method to some well-known nonlinear parabolic equations that appear in many scientific applications. The work also confirmed the power of the Haar wavelet method in handling nonlinear equations in general. This method can be easily extended to find the solution of all other non-linear parabolic equations. Another benefit of our method is that the scheme presented here, with some modifications, seems to be easily extended to solve model equations including more mechanical, physical or biophysical effects, such as nonlinear convection, reaction, linear diffusion and dispersion. The complexity with respect to 3 dimensional spatial variable for solving other nonlinear parabolic problems can be solved easily. Use of Haar wavelets for solving other type of nonlinear parabolic equation is presently ongoing.

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